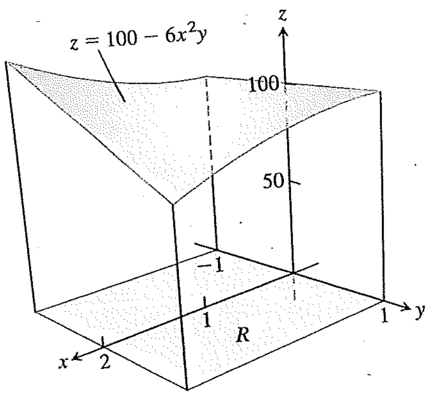

FIGURE 15.6 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 1).

Solution Figure 15.6 displays the volume beneath the surface. By Fubini's Theorem,

$$\iint_R f(x, y)\,dA = \int_{-1}^{1}\int_{0}^{2}(100 - 6x^2y)\,dx\,dy = \int_{-1}^{1}\Big[100x - 2x^3y\Big]_{x=0}^{x=2}\,dy$$

$$= \int_{-1}^{1}(200 - 16y)\,dy = \Big[200y - 8y^2\Big]_{-1}^{1} = 400.$$

Reversing the order of integration gives the same answer:

$$\int_0^2\int_{-1}^{1}(100 - 6x^2y)\,dy\,dx = \int_0^2\Big[100y - 3x^2y^2\Big]_{y=-1}^{y=1}\,dx$$

$$= \int_0^2\big[(100 - 3x^2) - (-100 - 3x^2)\big]\,dx$$

$$= \int_0^2 200\,dx = 400.$$ ■

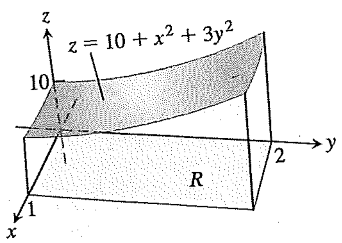

FIGURE 15.7 The double integral $\iint_R f(x, y)\,dA$ gives the volume under this surface over the rectangular region R (Example 2).

EXAMPLE 2 Find the volume of the region bounded above by the elliptical paraboloid $z = 10 + x^2 + 3y^2$ and below by the rectangle R: $0 \le x \le 1, 0 \le y \le 2$.

Solution The surface and volume are shown in Figure 15.7. The volume is given by the double integral

$$V = \iint_R (10 + x^2 + 3y^2)\,dA = \int_0^1\int_0^2 (10 + x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1\Big[10y + x^2y + y^3\Big]_{y=0}^{y=2}\,dx$$

$$= \int_0^1 (20 + 2x^2 + 8)\,dx = \Big[20x + \frac{2}{3}x^3 + 8x\Big]_0^1 = \frac{86}{3}.$$ ■

Exercises 15.1

Evaluating Iterated Integrals

In Exercises 1–14, evaluate the iterated integral.

1. $\int_1^2\int_0^4 2xy\,dy\,dx$
2. $\int_0^2\int_{-1}^1 (x - y)\,dy\,dx$
3. $\int_{-1}^0\int_{-1}^1 (x + y + 1)\,dx\,dy$
4. $\int_0^1\int_0^1 \left(1 - \frac{x^2 + y^2}{2}\right)dx\,dy$
5. $\int_0^3\int_0^2 (4 - y^2)\,dy\,dx$
6. $\int_0^3\int_{-2}^0 (x^2y - 2xy)\,dy\,dx$
7. $\int_0^1\int_0^1 \frac{y}{1 + xy}\,dx\,dy$
8. $\int_1^4\int_0^4 \left(\frac{x}{2} + \sqrt{y}\right)dx\,dy$
9. $\int_0^{\ln 2}\int_1^{\ln 5} e^{2x+y}\,dy\,dx$
10. $\int_0^1\int_1^2 xye^x\,dy\,dx$
11. $\int_{-1}^2\int_0^{\pi/2} y\sin x\,dx\,dy$
12. $\int_\pi^{2\pi}\int_0^\pi (\sin x + \cos y)\,dx\,dy$
13. $\int_1^4\int_1^e \frac{\ln x}{xy}\,dx\,dy$
14. $\int_{-1}^2\int_1^2 x\ln y\,dy\,dx$

Evaluating Double Integrals over Rectangles

In Exercises 15–22, evaluate the double integral over the given region R.

15. $\iint_R (6y^2 - 2x)\,dA$, $\quad R$: $0 \le x \le 1, \quad 0 \le y \le 2$
16. $\iint_R \left(\frac{\sqrt{x}}{y^2}\right)dA$, $\quad R$: $0 \le x \le 4, \quad 1 \le y \le 2$
17. $\iint_R xy\cos y\,dA$, $\quad R$: $-1 \le x \le 1, \quad 0 \le y \le \pi$
18. $\iint_R y\sin(x + y)\,dA$, $\quad R$: $-\pi \le x \le 0, \quad 0 \le y \le \pi$

19. $\iint_R e^{x-y}\,dA, \quad R: \; 0 \le x \le \ln 2, \; 0 \le y \le \ln 2$

20. $\iint_R xye^{xy^2}\,dA, \quad R: \; 0 \le x \le 2, \; 0 \le y \le 1$

21. $\iint_R \frac{xy^3}{x^2+1}\,dA, \quad R: \; 0 \le x \le 1, \; 0 \le y \le 2$

22. $\iint_R \frac{y}{x^2y^2+1}\,dA, \quad R: \; 0 \le x \le 1, \; 0 \le y \le 1$

In Exercises 23 and 24, integrate f over the given region.

23. **Square** $f(x, y) = 1/(xy)$ over the square $1 \le x \le 2$, $1 \le y \le 2$

24. **Rectangle** $f(x, y) = y \cos xy$ over the rectangle $0 \le x \le \pi$, $0 \le y \le 1$

25. Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the square $R: -1 \le x \le 1$, $-1 \le y \le 1$.

26. Find the volume of the region bounded above by the elliptical paraboloid $z = 16 - x^2 - y^2$ and below by the square $R: 0 \le x \le 2, 0 \le y \le 2$.

27. Find the volume of the region bounded above by the plane $z = 2 - x - y$ and below by the square $R: 0 \le x \le 1$, $0 \le y \le 1$.

28. Find the volume of the region bounded above by the plane $z = y/2$ and below by the rectangle $R: 0 \le x \le 4, 0 \le y \le 2$.

29. Find the volume of the region bounded above by the surface $z = 2 \sin x \cos y$ and below by the rectangle $R: 0 \le x \le \pi/2$, $0 \le y \le \pi/4$.

30. Find the volume of the region bounded above by the surface $z = 4 - y^2$ and below by the rectangle $R: 0 \le x \le 1$, $0 \le y \le 2$.

31. Find a value of the constant k so that $\int_1^2 \int_0^3 kx^2y\,dx\,dy = 1$.

32. Evaluate $\int_{-1}^{1} \int_0^{\pi/2} x \sin \sqrt{y}\,dy\,dx$.

33. Use Fubini's Theorem to evaluate

$$\int_0^2 \int_0^1 \frac{x}{1+xy}\,dx\,dy.$$

34. Use Fubini's Theorem to evaluate

$$\int_0^1 \int_0^3 xe^{xy}\,dx\,dy.$$

T 35. Use a software application to compute the integrals

a. $\int_0^1 \int_0^2 \frac{y-x}{(x+y)^3}\,dx\,dy$

b. $\int_0^2 \int_0^1 \frac{y-x}{(x+y)^3}\,dy\,dx$

Explain why your results do not contradict Fubini's Theorem.

36. If $f(x, y)$ is continuous over $R: a \le x \le b, c \le y \le d$ and

$$F(x, y) = \int_a^x \int_c^y f(u, v)\,dv\,du$$

on the interior of R, find the second partial derivatives F_{xy} and F_{yx}.

15.2 Double Integrals over General Regions

In this section we define and evaluate double integrals over bounded regions in the plane which are more general than rectangles. These double integrals are also evaluated as iterated integrals, with the main practical problem being that of determining the limits of integration. Since the region of integration may have boundaries other than line segments parallel to the coordinate axes, the limits of integration often involve variables, not just constants.

Double Integrals over Bounded, Nonrectangular Regions

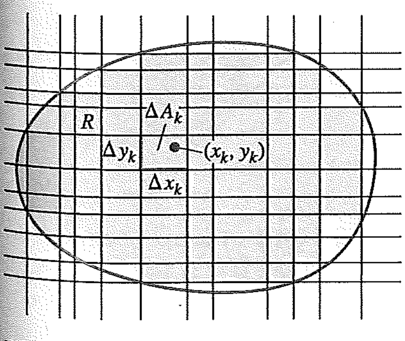

FIGURE 15.8 A rectangular grid partitioning a bounded, nonrectangular region into rectangular cells.

To define the double integral of a function $f(x, y)$ over a bounded, nonrectangular region R, such as the one in Figure 15.8, we again begin by covering R with a grid of small rectangular cells whose union contains all points of R. This time, however, we cannot exactly fill R with a finite number of rectangles lying inside R, since its boundary is curved, and some of the small rectangles in the grid lie partly outside R. A partition of R is formed by taking the rectangles that lie completely inside it, not using any that are either partly or completely outside. For commonly arising regions, more and more of R is included as the norm of a partition (the largest width or height of any rectangle used) approaches zero.

Once we have a partition of R, we number the rectangles in some order from 1 to n and let ΔA_k be the area of the kth rectangle. We then choose a point (x_k, y_k) in the kth rectangle and form the Riemann sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k.$$